JULY 9th 2011 PAPER 425/1

Algebra

1. Solve for x : $\frac{(1-x)^2}{2-x^2} = \frac{(1-a)^2}{2-a^2}$

$$\frac{1-2x+x^2}{2-x^2} = \frac{1-2a+a^2}{2-a^2}$$

$$(1-2x+x^2)(2-a^2) = (1-2a+a^2)(2-x^2)$$

$$2-a^2-4x+2a^2x+2x^2-a^2x^2 = 2-4a+2a^2-x^2+2ax^2-a^2x^2$$

$$4(x-a)-3(x^2-a^2)+2ax(x-a)=0$$ factorise by grouping

$$(x-a)[4-3(x+a)+2ax]=0$$

Either $x=a$ Or $4-3x-3a+2ax=0$

So, $x = \frac{3a-4}{2a-3}$

2. Given that the first three terms in the expansion in ascending powers of x of $(1-8x)^{1/4}$ are the same as the first three terms in the expansion of $\left(\frac{1+ax}{1+bx}\right)$, find the values of a and b.

Hence, find an approximation to $(0.6)^{1/4}$ in the form p/q.

$$(1-8x)^{1/4} = 1+\frac{1}{4}(-8x)+\frac{1}{2!}.\frac{1}{4}.\frac{-3}{4}(-8x)^2+\ldots$$

$$= 1-2x-6x^2-\ldots$$

$$\frac{1+ax}{1+bx} = (1+ax)\{1-bx+b^2x^2+..\}$$

$= 1+(a-b)x+(b^2-ab)x^2+\ldots$

Since the first three terms of the expansion are the same.

$a-b=-2, b^2-ab=-6$

Hence b = -3 and a = -5

Thus; $(1-8x)^{1/4} \approx \dfrac{1-5x}{1-3x}$

Substituting $x=0.05,$ we have

$(1-0.4)^{1/4} \approx \dfrac{1-0.25}{1-0.15} = \dfrac{0.75}{0.85}$ hence $(0.6)^{1/4} \approx 15/17$

4. Given $x \rangle 2$ and $\left(\dfrac{x+2}{x}\right)^{-1/2} = a+\dfrac{b}{x}+\dfrac{c}{x^2}+\dfrac{d}{x^3}+-+-$, use the binomial expansion and determine the values of a, b, c, d. By taking $x=100$, find the approximate value of $\left(\dfrac{450}{51}\right)^{1/2}$ correct to 4 decimal places.

$$\left(1+\frac{2}{x}\right)^{-1/2} = 1+-\frac{1}{2}\left(\frac{2}{x}\right)+\frac{-\frac{1}{2}.-\frac{3}{2}}{2!}\left(\frac{2}{x}\right)^2+\frac{-\frac{1}{2}.-\frac{3}{2}.-\frac{5}{2}}{3!}\left(\frac{2}{x}\right)^3+\ldots$$

$= 1-\dfrac{1}{x}+\dfrac{3}{2x^2}-\dfrac{5}{2x^3}+-\ldots$, hence, $a=1,\ b=-1,\ c=\frac{3}{2},\ d=-\frac{5}{2}$

When $x=100$

$$\left(\frac{100+2}{100}\right)^{-1/2} = \left(\frac{102}{100}\right)^{1/2} = \left(\frac{51}{50}\right)^{1/2}$$

So, to estimate $\left(\dfrac{450}{51}\right)^{1/2} = \left(\dfrac{9\times 50}{51}\right)^{1/2} = 3\left(\dfrac{50}{51}\right)^{1/2} = \dfrac{1}{3}\left(\dfrac{51}{50}\right)^{-1/2}$

Thus, $\dfrac{1}{3}\left(\dfrac{51}{50}\right)^{-1/2} = 1-\dfrac{1}{100}+\dfrac{3}{20000}-\dfrac{5}{2000000}$

$$\therefore \left(\frac{51}{50}\right)^{-1/2} = 3(1 - \frac{1}{100} + \frac{3}{20000} - \frac{5}{2000000})$$

$= \frac{1980295}{2000000} x\, 3 = 2.9704425 \approx 2.9704$ correct to 4 d.p.

5. Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all $n \geq 1$.

Let $a_n = 8^n - 7n + 6$

For $n = 1, \quad a_1 = 8 - 7 + 6 = 7$, divisible by 7.

For $n = 2, \quad a_1 = 64 - 14 + 6 = 56$, divisible by 7.

For $n = k, \quad a_k = 8^k - 7k + 6$,

When $n = k+1, \quad a_{k+1} = 8^{k+1} - 7(k+1) + 6$

Thus $a_{k+1} - a_k = 8^{k+1} - 7(k+1) + 6 - \left(8^k - 7k + 6\right)$

$= 8^k.8 - 8^k - 7k + 7k - 7 + 6 - 6$

$= 7.8^k - 7 = 7\left(8^k - 1\right)$ which is still divisible by 8. Thus if its true for $n = 1, 2, \ldots, k, k+1$, then $8^n - 7n + 6$ is divisible by 7 for all $n \geq 1$.

6. Find the values of λ for which $10x^2 + 4x + 1 = 2\lambda x(2 - x)$ has equal roots.

$10x^2 + 4x + 1 = 4\lambda x - 2\lambda x^2 \quad \Leftrightarrow \quad (10 + 2\lambda)x^2 + (4 - 4\lambda)x + 1 = 0$

For equal roots: $b^2 - 4ac = 0$, so $(4 - 4\lambda)^2 - 4(10 + 2\lambda) = 0$

$16\lambda^2 - 40\lambda - 24 = 0 \quad \Leftrightarrow \quad 2\lambda^2 - 5\lambda - 3 = 0$

$(\lambda - 3)(2\lambda + 1) = 0 \quad \therefore \quad \lambda = 3, \lambda = -\frac{1}{2}$

ALT; Let the roots be α & α

$2\alpha = \frac{-(4 - 4\lambda)}{10 + 2\lambda}$ where $\alpha = \frac{\lambda - 1}{\lambda + 5}$ and $\alpha^2 = \frac{1}{10 + 2\lambda}$

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So, $\left(\frac{\lambda-1}{\lambda+5}\right)^2 = \frac{1}{2(\lambda+5)}$, to get $2\lambda^2-5\lambda-3=0$

$\therefore \quad \lambda = 3, \lambda = -\frac{1}{2}$

7. Given that $z+2i$ is a factor of $z^4+2z^3+7z^2+8z+12$, solve the equation $z^4+2z^3+7z^2+8z+12=0$.

If $z+2i$ is a factor, then, $z-2i$, thus the quadratic factor is z^2+4

$$\begin{array}{r l} & z^2+2z+3 \\ \text{By long division, } z^2+4 & \overline{\big)\, z^4+2z^3+7z^2+8z+12} \\ & z^4 + \quad 4z^2 \end{array}$$

Thus, $(z^4+4)(z^2+2z+3)=0$

$z=\pm 2i, \quad z=\frac{-2\pm\sqrt{-8}}{2}$

The roots are $2i,\ -2i,\ -1+i\sqrt{2},\ -1-i\sqrt{2}$

NOTE: Candidate should differentiate between **a root** and **a factor**.

8. If $z=\cos\theta+i\sin\theta$, solve the equation $z^{4/3}=i$.

To solve $z^{4/3}=i$, we have $z^4=i^3=-i$

$z^4=\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi$

$=\cos-\frac{1}{2}\pi+i\sin-\frac{1}{2}\pi$

$=\cos-\frac{1}{2}\pi-i\sin\frac{1}{2}\pi$

$z=(\cos-\frac{1}{2}\pi+i\sin-\frac{1}{2}\pi)^{1/4}$

By De moivres thorem,

$$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{4}\right) - i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{4}\right) \text{ for } k = 0, 1, 2, 3$$

When $k = 0,\quad z_1 = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8} = 0.9239 - 0.3827i$

$k = 1,\quad z_2 = \cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8} = -0.3827 - 0.9239i$

$k = 2,\quad z_3 = \cos\frac{9\pi}{8} - i\sin\frac{9\pi}{8} = -0.9239 + 0.3827i$

$k = 3,\quad z_4 = \cos\frac{13\pi}{8} - i\sin\frac{13\pi}{8} = 0.3827 + 0.9239i$

ANALYSIS

1. Evaluate: $\int_0^{\frac{a}{2}} x^2\sqrt{(a^2 - x^2)}\,dx$

$$\int_0^{\frac{a}{2}} x^2\sqrt{(a^2 - x^2)}\,dx = \int_0^{\frac{a}{2}} x^2\sqrt{a^2\left(1 - \frac{x^2}{a^2}\right)}\,dx$$

Let $\frac{x}{a} = \sin\theta : dx = a\sin\theta\, d\theta$

$x = a\sin\theta$

$$= \int_0^{\frac{a}{2}} a.(a\sin\theta)^2\sqrt{(1 - \sin^2\theta)}.\, a\sin\theta\, d\theta$$

$x = 0,\quad \theta = 0$

$x = \frac{a}{2},\ \theta = \frac{\pi}{6}$

$$= a^4\int_0^{\frac{\pi}{6}} \sin^2\theta\cos^2\theta\, d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^4}{4}\int_0^{\frac{\pi}{6}} \sin^2 2\theta\, d\theta$$

$\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$

$$= \frac{a^4}{8}\int_0^{\frac{\pi}{6}} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{a^4}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{a^4}{8}\left[\left(\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right) - 0\right]$$

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$$= \frac{a^4}{8}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$$

$$= \frac{a^4}{16}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

2. Using the substitution

i) $y = e^x + 1$, find $\int \frac{dx}{(e^x+1)^2}$

$\int \frac{dx}{(e^x+1)^2}$ $\qquad$ $dy = e^x \ dx$

$\int \frac{dy}{y^2(y-1)}$, so let $\frac{1}{y^2(y-1)} \equiv \frac{A}{y} + \frac{B}{y^2} + \frac{C}{(y-1)}$

Thus $1 \equiv Ay(y-1) + B(y-1) + Cy^2$

Solving: $A = -1, \ B = -1, \ C = 1$

$$\int \frac{dy}{y^2(y-1)} = \int \frac{-1}{y}dy + \int \frac{-1}{y^2}dy + \int \frac{1}{(y-1)}dy$$

$$= -Iny + \frac{1}{y} + In(y-1) + c$$

$$= In\frac{y-1}{y} + \frac{1}{y} + c$$

$$= In\frac{e^x}{e^x+1} + \frac{1}{e^x+1} + c$$

ii) $t = \tan\frac{1}{2}x$, find the value of $\int_0^{\frac{\pi}{2}} \frac{dx}{3+5\cos x}$

$\int_0^{\frac{\pi}{2}} \frac{dx}{3+5\cos x}$ $\qquad$ $\cos x = \frac{1-t^2}{1+t^2}$, $t = \tan\frac{1}{2}x, \ dt = \frac{1}{2}\sec^2\frac{1}{2}xdx$

$$\int_0^1 \frac{(1+t^2)}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$ $x = \frac{\pi}{2},\ t = 1$

$x = 0,\ t = 0$

$$\int_0^1 \frac{1}{4-t^2}\,dt$$ let $\frac{1}{4-t^2} \equiv \frac{A}{2+t} + \frac{B}{2-t}$

$$\frac{1}{4}\int_0^1 \left(\frac{1}{2+t} + \frac{1}{2-t} \right) dt$$

$$\frac{1}{4}\left[In\frac{2+t}{2-t} \right]_0^1 = \frac{1}{4}[In3 - In1] = \frac{1}{4}In3$$

iii) $t = \tan x$ find $\int \frac{dx}{\cos^2 x + 4\sin^2 x}$

$$\int \frac{dx}{\cos^2 x + 4\sin^2 x} = \int \frac{\sec^2 x dx}{1 + 4\tan^2 x}$$ $t = \tan x,\ dt = \sec^2 x dx$

$$= \int \frac{\sec^2 x}{1+4t^2} \cdot \frac{dt}{\sec^2 x}$$

$$= \int \frac{dt}{1+4t^2}$$

$$= \frac{1}{2}\tan^{-1} 2t + c$$

$$= \frac{1}{2}\tan^{-1} 2(\tan x) + c$$

iv) $t = e^x$ to find $\int \frac{e^x}{e^x + e^{-x}}\,dx$

3. Find the equation of a curve whose gradient is given by xe^{x-y} and passes through the point $(0, In2)$.

$\frac{dy}{dx} = xe^{(x-y)}$, separating the variables: i.e $\frac{dy}{dx} = \frac{xe^x}{e^y}$

$\int e^y dy = \int xe^x dx$ let $u = x,\ \frac{dv}{dx} = e^x$; $\frac{du}{dx} = 1,\ v = e^x$

$\therefore e^y = xe^x - \int e^x\ dx$

$\therefore e^y = xe^x - e^x + C,\ \ x = 0,\ \ y = In2$

Thus; $e^{In2} = 0e^o - e^o + C,\quad C = 3$

$\therefore e^y = xe^x - e^x + 3$, thus $y = In(xe^x - e^x + 3)$

6. Solve the differential equations:

i) $2y(x+1)\frac{dy}{dx} = 4 + y^2$, given that $y = 2$ when $x = 3$.

Separating the variables,

$\int \frac{2y}{4+y^2} dy = \int \frac{dx}{x+1}$

$In(4+y^2) = In(x+1) + Ink \quad y = 2$ when $x = 3$

$In8 = In4 + Ink,\quad In2 = Ink,\quad k = 2$

$In(4+y^2) = In2(x+1)$ thus, $(4+y^2) = 2(x+1)$

To get $y = \sqrt{2(x-1)}$

ii) $y\cos^2 x\frac{dy}{dx} = \tan x + 2$, given that $y = 2$ when $x = \frac{\pi}{4}$

$$\int y\,dy = \int \frac{\tan x + 2}{\cos^2 x}\,dx$$

$\int y\,dy = \int \tan x \sec^2 x\,dx + 2\int \sec^2 x\,dx$, but $d(\tan x) = \sec^2 x\,dx$

$$\frac{y^2}{2} = \frac{\tan^2 x}{2} + 2\tan x + c$$

given that $y = 2$ when $x = \frac{\pi}{4}$

$2 = \frac{1}{2} + 2 + c$ thus, $c = -\frac{1}{2}$

$$\frac{y^2}{2} = \frac{\tan^2 x}{2} + 2\tan x - \frac{1}{2},$$

Thus $y^2 = \tan^2 x + 4\tan x - 1$

7. Sketch the curve $y = \dfrac{x^3}{9 - x^2}$, clearly stating the asymptotes.

Intercepts: when $y = 0, \quad x = 0$ so, $(0, \ 0)$, thus curve crosses the axes at origin.

Vertical asymptotes: $9 - x^2 = 0$, so vertical asymptotes at $x = -3, \quad x = 3$

To find the slant asymptotes: we perform long division:

$$\begin{array}{r l} & -x \\ -x^2 - 9 \big) & x^3 + 0x^2 + 0x + 0 \\ & -\left(x^3 - 9x\right) \\ \hline & \quad 9x \end{array}$$

We get $y = -x + \dfrac{9x}{9 - x^2}$, thus as $x \to \pm\infty, \quad y \to -x$, so the line $y = -x$ is the slant asymptote.

For the turning points, we have $\frac{dy}{dx}=0$,

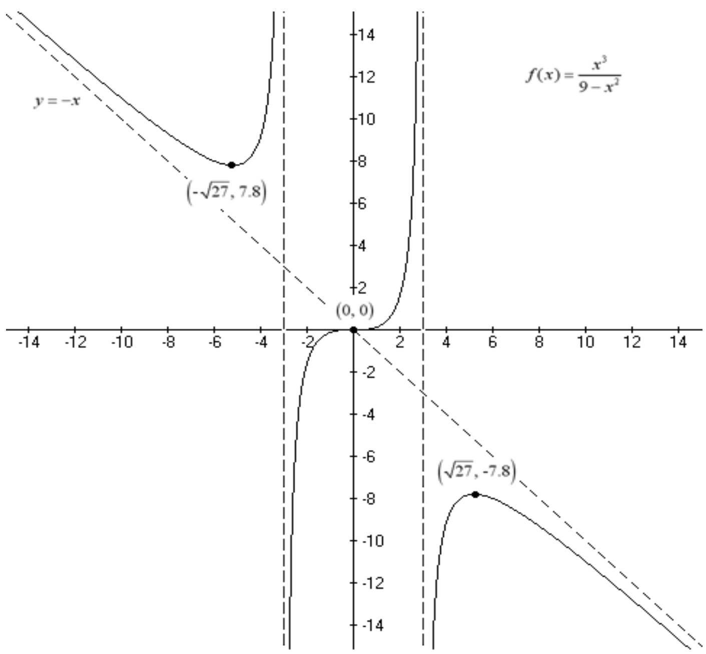
$$\frac{dy}{dx}=\frac{3x^2\left(9-x^2\right)-x^3.-2x}{\left(9-x^2\right)^2}=0$$

$$27x^2-3x^4+2x^4=0$$

$x^2\left(27-x^2\right)=0$ these occur when $x=0$, $x=\pm\sqrt{27}$,

$x=0,\ y=0$, $x=3\sqrt{3},\ y=-7.794$, $x=-3\sqrt{3},\ y=7.794$

i.e curve has a local minimum at $(-5.196\,,\,7.794)$ and a local maximum at $(5.196\,,\,-7.794)$ and a negative point of inflexion at $(0,\ \ 0)$



8. Given that $y=In\left(x^2+2x+3\right)$, show that $\frac{d^3y}{dx^3}\left(x^2+2x+3\right)+(4x+4)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$, hence, find Maclaurin's expansion of y showing the first four non-zero terms and approximate $In3.21$ correct to four decimal places.

$$y = In\left(x^2 + 2x + 3\right) \qquad \frac{dy}{dx} = \frac{2x+2}{x^2 + 2x + 3}$$

$$\left(x^2 + 2x + 3\right)\frac{dy}{dx} = 2x + 2$$

$$\left(x^2 + 2x + 3\right)\frac{d^2y}{dx^2} + (2x+2)\frac{dy}{dx} = 2$$

$$\left(x^2 + 2x + 3\right)\frac{d^3y}{dx^3} + (2x+2)\frac{d^2y}{dx^2} + \left(2x+2\right)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$\therefore \; \frac{d^3y}{dx^3}\left(x^2 + 2x + 3\right) + \left(4x+4\right)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

Let $f\left(x\right) = In\left(x^2 + 2x + 3\right)$

$$f\left(0\right) = In3 \approx 1.0986$$

$$f'\left(0\right) = \frac{2}{3}$$

$$f''\left(0\right) = \frac{2 - 2\times\frac{2}{3}}{3} = \frac{2}{9}$$

$$f'''\left(0\right) = \frac{2 - 2\times\frac{2}{3} - 2\times\frac{2}{9} - 2\times\frac{2}{9}}{3} = -\frac{20}{7}$$

$$\Rightarrow \; f\left(x\right) = y = In3 + \frac{2}{3}x + \frac{2}{9}.\frac{x^2}{2!} + -\frac{20}{27}.\frac{x^3}{3!} + \ldots$$

$$= In3 + \frac{2}{3}x + \frac{x^2}{9} - \frac{10}{81}x^3 + \ldots$$

For $In3.21$ take $x = 0.1$

$$In3.21 \approx In3 + \frac{2}{3}\times 0.1 + \frac{0.1^2}{9} - \frac{10}{81}\times 0.1^3$$

$$= 1.16627$$

≈ 1.1663 correct to 4 d.p.

9. The volume V of a cone is given by $V = \frac{1}{3}\pi r^2 h$. If the volume increases by $\frac{2}{3}\pi cm^3 / \min$ and height increases by 0.03 $cm / \min$, find the rate of change of the radius when $r = 10cm$ and $h = 5cm$.

$V = \frac{1}{3}\pi r^2 h$ depends on r and t

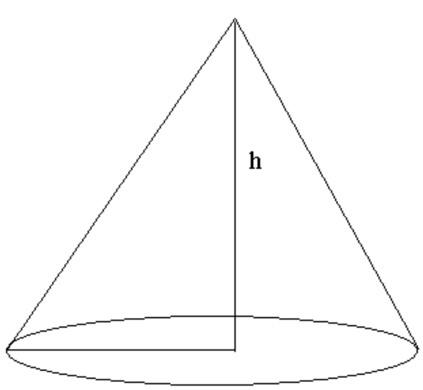
$$\Rightarrow \frac{dv}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt} + \frac{\pi}{3} h.2r \frac{dr}{dt}$$

$$\Leftrightarrow \frac{2\pi}{3} = \frac{\pi}{3}(100x0.03 + 2x10x5\frac{dr}{dt})$$

We get $2 - 3 = 100\frac{dr}{dt}$

$$\therefore \frac{dr}{dt} = -0.01cm / \min$$

10. Sand falls on to horizontal ground at the rate of 9m^3 per minute and forms a heap in the shape of a right circular cone with vertical angle 120^o. Show that $20\sqrt{3}$ seconds after sand begins to fall, the rate which the radius of the base of the pile is increasing is $\frac{\sqrt{3}}{\pi^{1/3}} m.\min^{-1}$.



$\frac{dV}{dt} = 9$, $V = \frac{1}{3}\pi r^2 h$

$$\frac{r}{h} = \tan 60^o \Rightarrow h = \frac{r}{\sqrt{3}} = \frac{\sqrt{3}}{3} r$$

$$\therefore V = \frac{\pi r^2}{3}.\frac{r}{\sqrt{3}} = \frac{\pi r^3}{3\sqrt{3}}$$

$$\frac{dr}{dt} = \frac{dV}{dt}.\frac{dr}{dV} = 9\frac{dr}{dV}$$

But $\frac{dV}{dr} = \frac{\pi r^2}{\sqrt{3}} \Rightarrow \frac{dr}{dV} = \frac{\sqrt{3}}{\pi r^2}$

$$\therefore \frac{dr}{dt} = 9.\frac{\sqrt{3}}{\pi r^2},$$

and since $\frac{dV}{dt} = 9$, then $V = 9t$

$\Rightarrow \frac{\pi r^3}{3\sqrt{3}} = \frac{9 \times 20\sqrt{3}}{60}$ thus $r^3 = 9x\frac{20\sqrt{3}}{60}x\frac{3\sqrt{3}}{\pi}$

$\therefore r = \frac{3}{\pi^{\frac{1}{3}}}$ $\quad \because \frac{dr}{dt} = \frac{9\sqrt{3}}{\pi}x\frac{1}{\frac{3^2}{\pi^{\frac{2}{3}}}}$ = $\frac{\sqrt{3}}{\pi^{\frac{1}{3}}}m.\min^{-1}$

GEOMETRY

1. Prove that if two circles $C_1 : x^2 + y^2 + 2gx + 2fy + c = 0$ and $C_2 : x^2 + y^2 + 2Gx + 2Fy + C = 0$ are orthogonal then $2gG + 2fF = c + C$.

For orthogonal circles, $d^2 = r_1^2 + r_2^2$

For $C_1 : x^2 + y^2 + 2gx + 2fy + c = 0$,

$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$ thus $C_1(-g, -f)$, $r_1^2 = g^2 + f^2 - c$

For $C_2 : x^2 + y^2 + 2Gx + 2Fy + C = 0$,

$(x+G)^2 + (y+F)^2 = G^2 + F^2 - C$ thus $C_2(-G, -F)$, $r_2^2 = G^2 + F^2 - C$

So, substitute in $d^2 = r_1^2 + r_2^2$

$$g^2 + f^2 - c + G^2 + F^2 - C = (-g+G)^2 + (-f+F)^2$$

$$g^2 + f^2 - c + G^2 + F^2 - C = g^2 - 2gG + G^2 + f^2 - 2fF + F^2$$

Then $2gG + 2fF = c + C$. as required.

2. Find the equation of the line through the origin and concurrent with $2x-5y=3$ and $3x-4y=-2$.

Let $2x-5y-3+\lambda(3x-4y+2)=0$, where λ is a constant to be found.

But $x=0,\ \ y=0$

$\Rightarrow -3+2\lambda=0$, $\lambda=\frac{3}{2}$ or $\frac{2}{3}$

$\Rightarrow\ 2x-5y-3+\frac{3}{2}(3x-4y+2)=0$

$4x-10y-6+9x-12y+6=0$

$13x\ -22y=0$ or $22y-13x=0$

ALT. Let $2x-5y=3$, $3x-4y=-2$, solving simultaneously,

We get $x=-\frac{22}{7}$, $y=-\frac{13}{7}$

Grad of line $\frac{0--\frac{13}{7}}{0--\frac{22}{7}}=\frac{13}{22}$

$\therefore\ \frac{y-0}{x-0}=\frac{13}{22}\quad 22y-13x=0$

OR Let $2x-5y-3=\lambda(3x-4y+2)$

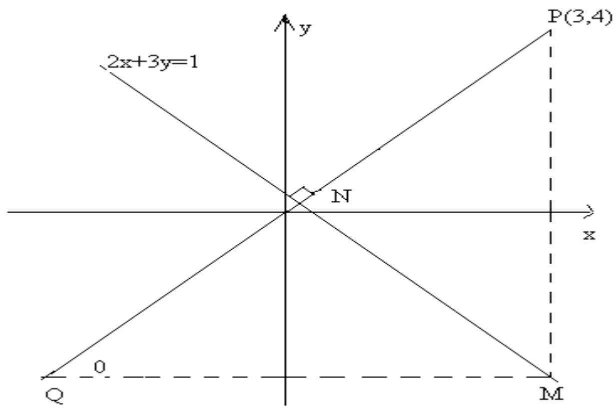
Put $x=0,\ \ y=0$

$\Rightarrow\ -3=2\lambda$, $\lambda=-\frac{3}{2}$ or $-\frac{2}{3}$

$2x-5y-3=-\frac{3}{2}(3x-4y+2)$

$22y-13x=0$

3. PN, the perpendicular from P(3, 4) to the line $2x+3y=1$ is produced to Q such that NQ = PN. Find the coordinates of Q.



Gradient of PQ = $\frac{3}{2}$ (perpendicular to $2x+3y=1$)

Equation of PQ is: $y-4=\frac{3}{2}(x-3)$ i.e $3x-2y-1$.

Solving this simultaneously with $2x+3y=1$, we obtain coordinates of N, $(\frac{5}{13},\frac{1}{13})$.

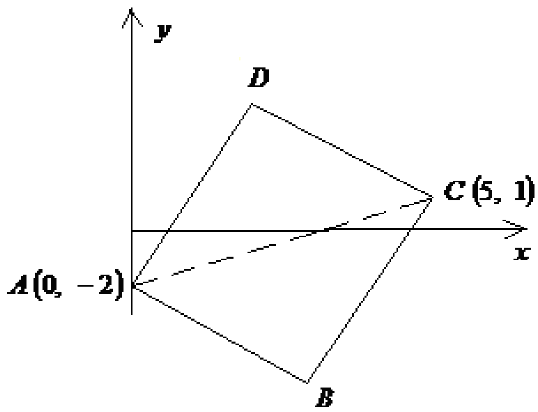
Let the coordinates of Q be (α,β), the coordinates of the midpoint of PQ i.e N are $\left(\frac{\alpha+3}{2},\frac{\beta+4}{2}\right)$

Thus; $\frac{\alpha+3}{2}=\frac{5}{13}$ giving $\alpha=-2\frac{3}{13}$

$\frac{\beta+4}{2}=\frac{1}{13}$ giving $\beta=-3\frac{11}{13}$

Coordinates of Q are $(-2\frac{3}{13},-3\frac{11}{13})$.

4. $ABCD$ is a square; A is the point $(0,\ -2)$ and C is the point $(5,\ 1)$, AC being the diagonal. Find the equations of the lines AB and BC.



AD & AB make angles of 45^o with diagonal AC .

So, the gradient of $AC = \frac{1-(-2)}{5-0} = \frac{3}{5}$, so if m_1 & m_2 are the gradients of AD & AB, then, $\tan 45 = \frac{m_1 - \frac{3}{5}}{1+\frac{3}{5}m_1}$, where $m_1 = 4$ and so $m_2 = -\frac{1}{4}$ since AD & AB are perpendicular.

Equation of AD is; $\frac{y+2}{x-0} = 4$, $y = 4x - 2$

Equation of AB is; $\frac{y+2}{x-0} = -\frac{1}{4}$, $4y + x + 8 = 0$

Trigonometry

1. Prove that. $\frac{\tan A - 3\tan 3A}{\cot A - 3\cot 3A} = \frac{\tan A - 3\cot A}{\cot A - 3\tan A}$

From L.H.S $\frac{\tan A - 3\tan 3A}{\cot A - 3\cot 3A} = \frac{\tan A - \frac{3(3\tan A - \tan^3 A)}{1 - 3\tan^2 A}}{\cot A - \frac{3(\cot^3 A - 3\cot A)}{3\cot^3 A - 1}}$

$$= \frac{\dfrac{\tan A - 3\tan^3 A - 9\tan A + 3\tan^3 A}{1 - 3\tan^2 A}}{\dfrac{3\cot^3 A - 3\cot^3 A - \cot A + 9\cot A}{3\cot^2 A - 1}}$$

$$= \frac{-8\tan A}{1 - 3\tan^2 A} \cdot \frac{3\cot^2 A - 1}{8\cot A}$$

$$= \frac{\tan A - 3\cot A}{\cot A - 3\tan A} \text{ as the R.H.S}$$

2. Prove that in a triangle ABC,

i) $\dfrac{a+b-c}{a+b+c} = \tan\tfrac{1}{2}A\tan\tfrac{1}{2}B$

R.H.S $\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)^2}{s^2(s-a)(s-b)}}$$

$$= \frac{s-c}{s}$$

$$= \frac{\frac{a+b+c}{2} - c}{\frac{a+b+c}{2}}$$

$$= \frac{a+b-c}{a+b+c} \text{ as L.H.S}$$

ALT: From L.h.S.

$$\frac{a+b-c}{a+b+c} = \frac{2R(\sin A + \sin B - \sin C)}{2R(\sin A + \sin B + \sin C)}$$

$$= \frac{2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + 2\cos\tfrac{(B+C)}{2}\sin\tfrac{(B-C)}{2}}{2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + 2\sin\tfrac{(B+C)}{2}\cos\tfrac{(B-C)}{2}}$$

$\cos\frac{B+C}{2} = \cos\left(90 - \frac{A}{2}\right) = \sin\tfrac{1}{2}A$

$\sin\frac{B+C}{2} = \cos\left(90 - \frac{A}{2} = \cos\tfrac{1}{2}A\right)$

$$= \frac{\sin\frac{1}{2}A\cos\frac{1}{2}A + \sin\frac{A}{2}\sin\frac{(B-C)}{2}}{\sin\frac{1}{2}A\cos\frac{1}{2}A + \cos\frac{A}{2}\cos\frac{(B-C)}{2}}$$

$$= \frac{\sin\frac{1}{2}A\left(\cos\frac{1}{2}A + \sin\frac{B-C}{2}\right)}{\cos\frac{1}{2}A\left(\sin\frac{1}{2}A + \cos\frac{B-C}{2}\right)}$$

$$= \frac{\sin\frac{1}{2}A\left(\sin\frac{B+C}{2} + \sin\frac{B-C}{2}\right)}{\cos\frac{1}{2}A\left(\cos\frac{B+C}{2} + \cos\frac{B-C}{2}\right)}$$

$$= \frac{\sin\frac{1}{2}A\left(2\sin\frac{B}{2}\cos\frac{C}{2}\right)}{\cos\frac{1}{2}A\left(2\cos\frac{B}{2}\cos\frac{C}{2}\right)}$$

$= \tan\frac{1}{2}A\tan\frac{1}{2}B$ as required.

ii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$

From the L.H.S

$$\sin^2 A + \sin^2 B + \sin^2 C = \frac{1}{2}(1-\cos 2A) + \frac{1}{2}(1-\cos 2B) + \sin^2 C$$

$$= 1 - \frac{1}{2}(\cos 2A - \cos 2B) + \left(1 - \cos^2 C\right)$$

$$= 2 - \frac{1}{2}(2\cos(A+B)\cos(A-B)) - \cos^2 C \quad : \cos(A+B) = -\cos C$$

$$= 2 - (-\cos C\cos(A-B)) - \cos^2 C$$

$$= 2 + \cos C(\cos(A-B) - \cos C)$$

$$= 2 + \cos C(\cos(A+B) + \cos(A-B))$$

$$= 2 + 2\cos A\cos B\cos C$$

3. Given that in a triangle MNB, $\tan A = \frac{n+b}{n-b}\tan\frac{1}{2}M$, where n, b & m are sides of the triangle MNB, show that $m = (n-b)\sec A\cos\frac{1}{2}M$.

$$\tan A = \frac{n+b}{n-b}\tan\frac{1}{2}M,$$

$$\tan^2 A = \frac{(n+b)^2}{(n-b)^2}\cdot\frac{\sin^2\frac{1}{2}M}{\cos^2\frac{1}{2}M}, \qquad \sec^2 A - 1 = \frac{(n+b)^2}{(n-b)^2}\cdot\frac{\sin^2\frac{1}{2}M}{\cos^2\frac{1}{2}M}$$

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$$(n-b)^2\cos^2\tfrac{1}{2}M\left(\sec^2 A-1\right)=(n+b)^2\sin^2\tfrac{1}{2}M$$

$$(n-b)^2\cos^2\tfrac{1}{2}M\sec^2 A=(n+b)^2\left(1-\cos^2\tfrac{1}{2}M\right)+(n-b)^2\cos^2\tfrac{1}{2}M$$

$$(n-b)^2\cos^2\tfrac{1}{2}M\sec^2 A=(n+b)^2-(n+b)^2\cos^2\tfrac{1}{2}M+(n-b)^2\cos^2\tfrac{1}{2}M$$

$$(n-b)^2\cos^2\tfrac{1}{2}M\sec^2 A=(n+b)^2+\cos^2\tfrac{1}{2}M\left((n+b)^2-(n-b)^2\right)$$

$$(n-b)^2\cos^2\tfrac{1}{2}M\sec^2 A=(n+b)^2-4nb\cos^2\tfrac{1}{2}M\ ,\qquad \cos^2\tfrac{1}{2}M=\tfrac{1}{2}(1+\cos M\ )$$

$$(n-b)^2\cos^2\tfrac{1}{2}M\sec^2 A=n^2+2nb+b^2-2nb\cos M-2bn$$

$$(n-b)^2\cos^2\tfrac{1}{2}M\sec^2 A=n^2+b^2-2nb\cos M$$

$$(n-b)^2\cos^2\tfrac{1}{2}M\sec^2 A=m^2$$

Thus $m=(n-b)\sec A\cos\tfrac{1}{2}M$

VECTORS

1. A and B are points whose position vectors are $a=2i+5k$, $b=i+3j+k$ respectively, and the equations of the line L are $\dfrac{x-3}{2}=\dfrac{y-2}{2}=\dfrac{z-2}{1}$. Determine

i) the equation of the plane π which contains A and is perpendicular to L, and verify that B lies in the plane π.

Equation of the plane is of the form $r\,.\,n=d$ or $ax+by+cz=d$

Vector normal to the plane is the direction vector of the line $n=2i+2j+k$

Thus $a\,.\,n=d$ is $(2i+0j+5k).(2i+2j+k)\ \therefore d=9$

Equation of the plane is $r\,.\,(2i+2j+k)=9$ or $2x+2y+z=9$

For the point B, $d=(i+3j+k).(2i+2j+k)=9$, thus since B satisfies the equation of the plane, then it lies there.

ii) the angle between the plane π above and the line $r=3i+j-3k+t\,(i-2j-4k)$.

Let θ be the angle between the plane and the line, thus the angle between the normal to the plane and the line is given by $n\,.\,d=|n|\,|d|\sin\theta$

Thus $n.d = (2i+2j+k).(i-2j-4k) = -6$

$|n| = \sqrt{4+4+1} = \sqrt{9} = 3$

$|d| = \sqrt{1+4+16} = \sqrt{21}$

$\sin\theta = \frac{-6}{3\sqrt{21}} = -\frac{2}{\sqrt{21}}$

2. Find the point of intersection of the lines $x = \frac{y+1}{2} = \frac{z-5}{-1}$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$

Let $x = \frac{y+1}{2} = \frac{z-5}{-1} = \lambda$, $\Rightarrow x = \lambda,\ y = 2\lambda - 1,\ z = -\lambda + 5$

Let $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1} = \mu \Rightarrow x = -\mu + 1,\ y = -3\mu - 3,\ z = \mu + 4$

Solve to obtain $\mu = -4,\ \lambda = 5$, check to see that they do intersect,

Hence the point is $(5, 9, 0)$.

b) Prove that the vectors $a = 3i + j - 4k$, $b = 5i - 3j - 2k$ and $c = 4i - j - 3k$ are coplanar.

For coplanar vectors a, b, c, $a = \lambda b + tc$, where $\lambda \,\&\, t$ are scalars.

So, $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} = \lambda \begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$, thus $\begin{matrix} 5\lambda + 4t = 3 \\ -3\lambda - t = 1 \\ -2\lambda - 3t = -4 \end{matrix}$ solving to get $t = 2,\ \lambda = -1$

Check using equation (iii) L.H.S = $2 + -6 = -4$ = R.H.S

Since the value of $\lambda \,\&\, t$ are consistent, then the vectors are coplanar.

c) The vector equation of a line is given by $r = 17i + 2j - 6k + \lambda(-9i + 3j + 9k)$. Find:

i) A vector parallel to this line.

Note: Parallel lines have the same direction ratios.

$-9:3:9=-3:1:3$, so either $-3i+j+3k$ or $3i-j-3k$ are parallel to the line.

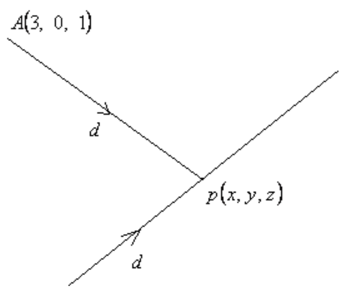
ii) State one point through which the line passes.

$(17, 2, -6)$

3i) Find the perpendicular distance of the point $(3,0,1)$ from the line $\frac{x-1}{3}=\frac{y+2}{4}=\frac{z}{12}$

$\frac{x-1}{3}=\frac{y+2}{4}=\frac{z}{12}$; $(3,\ 0,\ 1)$

so let $\frac{x-1}{3}=\frac{y+2}{4}=\frac{z}{12}=\lambda$, $x=1+3\lambda,\ y=4\lambda-2,\ z=12\lambda$



$$AP=OP-OA$$

$$=\begin{pmatrix}3\lambda+1\\4\lambda-2\\12\lambda\end{pmatrix}-\begin{pmatrix}3\\0\\1\end{pmatrix}=\begin{pmatrix}3\lambda-2\\4\lambda-2\\12\lambda-1\end{pmatrix}$$

AP is perpendicular to $d=3i+4j+12k$, thus, $AP\,.\,d=0$

$$\begin{pmatrix}3\lambda+1\\4\lambda-2\\12\lambda\end{pmatrix}.\begin{pmatrix}3\\4\\12\end{pmatrix}=0,\quad 9\lambda-6+16\lambda-8+144\lambda-12=0,\quad \lambda=\frac{2}{13}$$

$AP = \begin{pmatrix} \frac{6}{13}-2 \\ \frac{8}{13}-2 \\ \frac{24}{13}-1 \end{pmatrix} = \begin{pmatrix} -\frac{20}{13} \\ -\frac{8}{13} \\ \frac{11}{13} \end{pmatrix}$, the perpendicular distance

$$d = |AP| = \sqrt{\left(-\frac{20}{13}\right)^2 + \left(-\frac{18}{13}\right)^2 + \left(\frac{11}{13}\right)^2} = 2.23607$$

ii) Find the angle between the line $x-1 = \frac{y-2}{0} = \frac{z-3}{3}$ and the plane $3x+2y+z=1$.

$d = i + 0j + 3k$, $3x+2y+z=1$; $n = 3i+2j+k$

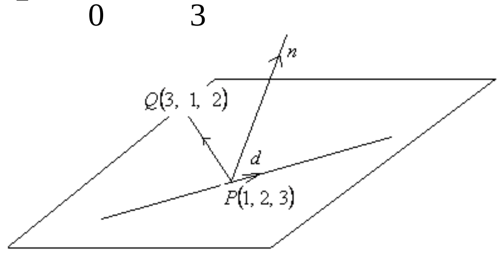
Let θ be the angle between the line and plane is given by $n.d = |n|\,|d|\sin\theta$

$n.d = (3i+2j+k).(i+0j+3k) = 6$

$|n| = \sqrt{1^2+3^2} = \sqrt{10}$, $|d| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$

$$\sin\theta = \frac{6}{\sqrt{10}\sqrt{14}} = \frac{6}{\sqrt{140}}, \quad \theta = 30.47^o$$

iii) Determine in dot product form the equation of a plane containing the point $(3,1,2)$ and the line $x-1 = \frac{y-2}{0} = \frac{z-3}{3}$



$PQ = (3i + j + 2k) - (i + 2j + 3k) = i - j - k$

Let the normal have direction ratios $a:b:c$

$n.PQ = 0$, thus, $(ai+bj+ck).(i-j-k) = a-b-c=0$

Also, $n.d=0$, $(ai+bj+ck).(i+0j+3k) = a+3c=0$

So, $a=-3c$, $b=-4c$, thus $a:b:c = -3c:-4c:c = -3:-4:1$

Equation is $r.\begin{pmatrix}-3\\-4\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}.\begin{pmatrix}-3\\-4\\1\end{pmatrix}$ thus $r.\begin{pmatrix}-3\\-4\\1\end{pmatrix} = -8$

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TRY:

1. The points A and B have coordinates $(2,\ 1,\ 1)$ and $(0,\ 5,\ 3)$ respectively. Find the equation of the line AB in parametric form. If C is the point $(5,\ -4,\ 2)$, find the coordinates of the point D on AB such that CD is perpendicular to AB.

2. Determine the equation of the plane through the points $A(1,\ 1,\ 2)$, $B(2,\ -1,\ 3)$ and $C(-1,\ 2,\ -2)$.

   b) A line through the point $D(-13,\ 1,\ 2)$ and parallel to the vector $12\mathbf{i}+6\mathbf{j}+3\mathbf{k}$ meets the plane in (a) at point E. Find:

   i) the coordinates of E.

   ii) The angle between the line and the plane.

3. Solve the simultaneous equations:
$$z_1 + z_2 = 8$$
$$4z_1 - 3iz_2 = 26 + 8i$$

4. Use De'Moivres theorem to find the modulus and argument of $\dfrac{(3+9i)^9(1+i)^3}{(-1-i)}$

5. Prove that in a triangle ABC,

   i) $a\cos ec\frac{1}{2}A = (b+c)\sec\frac{1}{2}(B-C)$, find C if $a = 5.3, b+c = 11.8, and A = 46^o$

   ii) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$, find the largest angle and area of the triangle in which $a = 10.6, b = 11.8, And\ c = 15.6$

THANK YOU AND KEEP WORKING HARD